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Piecewise Algebraic Functions

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This paper contains a thorough study of piecewise algebraic sets and functions. They are built up using finitely many parts, each of which is defined by finitely many algebraic equations or inequalities. They occur naturally in many concrete situations which are demonstrated by applications in linear algebra, complexity theory, and meromorphic differential equations. We also emphasize that in many cases effective procedures are given for the calculation of the various objects.

1. INTRODUCTION

In this paper we introduce a new concept, piecewise algebraic functions (PA-functions), which is motivated by various examples in different fields of mathematics:

Suppose we want to calculate a Jordan canonical form of a matrix with complex entries in a way which explains how the answer depends upon these entries when we vary them arbitrarily. Since there are finitely many choices for the result we deal in this case with a multi-valued "function"; moreover, we have to distinguish several different situations so that the domain of our "function" splits into several parts; on each of these parts, however, there exists a universal procedure leading to the desired result. Here it is important to recognize and describe the structure of these parts and the corresponding universal procedures.

Suppose we want to determine the algebraic complexity of a polynomial with fixed degree and want to know how this integer depends upon the coefficients of that polynomial if we vary them arbitrarily. Here we deal with a single-valued function which takes on only finitely many values and the problem really is to describe the set of all polynomials with a given complexity. We will show that such a set splits into finitely many parts, each of which can be described by universal polynomial conditions (involv-

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ing equations and inequalities) in terms of the coefficients of our polynomial. These parts will be called P-sets and play a central role in our theory.

We often consider a variable vector $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ restricted only by an inequality $g(x) \neq 0$, where g is a fixed non-trivial polynomial with integral coefficients. This is our concept of "independent variables" and could be viewed as an analytic counterpart to the algebraic concept of indeterminants. We also consider equations of the form $g(x) = 0$ with $g \in \mathbb{Z}[x]$ and the corresponding x-set of solutions. These sets generate a set algebra, whose elements will be called PA-sets (PA = piecewise algebraic). They are very similar to the constructible sets of Mumford [8, p. 37], except that he allows polynomials with arbitrary complex coefficients. So our PA-sets form a much smaller class which is more universal in nature, and these are typically the sets which we find in applications. For example, the domain of a PA-function is to consist of several disjoint parts of this type. On each of these parts the universal nature of a PA-function is described by considering its graph, which we require to be a PA-set in the product space. An example is the Jordan canonical form mentioned above.

It is the central theme of our discussion to analyze how certain mathematical objects depend upon their defining parameters. A rather involved case is the question of how the formal solutions of a meromorphic differential equation depend upon the coefficients of this equation. A basic problem, on the other hand, is the question of how the solutions of several algebraic equations depend upon their coefficients. In this context we single out a situation which has an especially simple structure: We assume that the variables $x \in \mathbb{C}^n$ are independent as explained before and that we have further variables $y = (y_1, \ldots, y_m) \in \mathbb{C}^m$, which are dependent in the sense that we have a polynomial equation in x and y_1 for the determination of y_1 (as a multi-valued function of x) followed by a polynomial equation in x, y_1, and y_2 for the determination of y_2 and so on, where additional requirements concerning these equations guarantee the solvability with a fixed number of distinct solutions. The resulting (x, y)-set will be called normal (for the precise definition the reader is referred to Section 5) and such a normal set represents a special complex manifold of complex dimension n.

These normal sets play the central role in our theory; in fact, we show that every PA-set can be decomposed into finitely many normal sets (Section 5, Decomposition Theorem). This is our main result with regards to the theory and has many applications. It is also closely connected with the Noether Normalization Lemma [8, p. 36]. For example, we derive, in this way, remarkable closure properties for our system of PA-functions: Besides arithmetic operations we can form the composition and inverse (if defined) without restrictions within the system; furthermore, image and preimage of PA-sets are PA-sets again. It turns out, that the PA-functions form the smallest system with these properties which contains the coordinate functions and meets certain trivial additional requirements. In this sense our PA-functions are the simplest functions which can have all these properties.

Of particular interest, of course, are the single-valued PA-functions. They have rational representations if we decompose their domains into suitable PA-sets and, accordingly, we call them PR-functions. The situation here is similar to Zariski's "Main Theorem" in algebraic geometry [4, p. 410; 8, p. 52]. We also refer the reader to [7], where we have already used this concept to analyze meromorphic equivalence of differential equations.

A rational function $r = p/q$ in the variables $x = (x_1, \dots, x_n)$ having integral coefficients can be viewed as an algebraic object in $\mathbb{Z}(x)$ or as an analytic object with independent variables restricted by the condition $q(x) \neq 0$. In the latter case we are interested in the evaluation of $r(x)$ for all x that make sense. It is natural to ask for a procedure to calculate $r(x)$ by means of arithmetic operations. Such a procedure should be effective and should not break down on the domain of r. In order to clarify the term "effective procedure" we take the pragmatic point of view, that means we will only use a list of "accepted" effective procedures. Usually an effective procedure will be given by some type of finite program which could, in principle, be executed by a computer, and it should be permitted to insert effective procedures into each other. The programs we think of at the present have a finite combinatorial structure which, in coded form, is described by various integers. We consider an *integer* as *given* if its sign and dyadic expansion are explicitly presented. In case that we only possess an accepted effective procedure which works out the explicit form of that integer, we call that integer *certainly computable* (cc). We like to emphasize that this means much more than well defined, because it is possible to define unique integers depending upon the solution of unsolved problems. So certain integers may become cc at the same time when our knowledge increases. Analogously, the combinatorial structure of a program may either be given explicitly or only be given in cc-form. Clearly cc programs are very desirable, and it is an important information about a well defined mathematical object if it can be worked out by means of a cc program. So we are going to discuss, e.g., rational functions which are cc and related cc programs which evaluate them by means of arithmetic operations. The cc point of view will be carried through the entire paper and is an essential improvement over statements of sheer existence.

Finally, we would like to mention some further consequences of our Decomposition Theorem: Given a PA-set in (x, y), we associate with each x the number $N(x)$ of related y. It turns out that the resulting cardinality function N must be a PA-function again (Cardinality Theorem, Section 6). In particular, the projection of the PA-set onto x (which is the x-set with $N(x) > 0$) must be a PA-set. This result is very similar to the projection theorem in algebraic geometry [8, p. 37]. It is interesting that we also have a cc-form of the Cardinality Theorem. Other finite-valued functions like rank or complexity behave very much like the cardinality function (see Section 10). Another interesting aspect of a PA-set is its dimension which can be interpreted algebraically or topologically or in the sense of measure theory leading to equivalent results (Section 7). These concepts can also be used for arbitrary sets, and PA-functions have the remarkable property that their application does not increase the dimension.

2. PA-Sets and PA-Functions

This section contains the definitions of our piecewise algebraic (PA) objects. All occurring sets belong to some $\mathbb{C}^n$ ($n \in \mathbb{N}$) if nothing else is said.

Definition. A set of the form $\{x \in \mathbb{C}^n\colon g(x) \neq 0,\ g_j(x) = 0 \text{ for } j = 1, \ldots, m\}$ with $m \in \mathbb{N}_0$, $g, g_j \in \mathbb{Z}[x]$ is called a *P-set* in the variable x. We say that it is *certainly computable* (cc) as soon as an effective procedure is *given* which completely computes m, n, and a possible choice of the polynomials g, g_j (i.e., of their degrees and coefficients which are integers) in finitely many steps. A *PA-set* consists of a finite union of P-sets. Such a PA-set M is named *certainly computable* (cc) as soon as an effective procedure is *given* for the determination of finitely many cc P-sets whose union yields M. A (cc) PA-set $F \subseteq \mathbb{C}^n \times \mathbb{C}^m$ ($n, m \in \mathbb{N}$) is called a (cc) *PA-relation* and we define $F^{-1} = \{(y, x)\colon (x, y) \in F\}$. Given a PA-relation F we say that x is the *preimage variable* and y is the *image variable*.

The P-sets are global sets of solutions of a system of polynomial equations with an inequality as restriction. Their unions, the PA-sets, differ from PA-relations by the fact that the latter ones exhibit a split of the variables.

Remark 1. *The PA-sets of $\mathbb{C}^n$ form a set algebra under the operations union ($\cup$), intersection ($\cap$) and difference ($\neg$). This algebra contains (e.g.) the sets $\varnothing, \mathbb{C}^n, \operatorname{diag}(\mathbb{C}^n) = \{(x, x)\colon x \in \mathbb{C}^{n/2}\}$ for even n, all of which are cc.* It is finitely generated by the sets $\{x \in \mathbb{C}^n\colon g(x) = 0\}$ with $g \in \mathbb{Z}[x]$. *Furthermore, the union, intersection, and the difference of cc PA-sets are again certainly computable.* The last statement is obvious for $M \cup M'$, since we use the effective procedure for M as well as the one for M'. In case of

$M \cap M'$ we first determine finitely many cc P-sets M_j, M'_k such that $M = \bigcup M_j$, $M' = \bigcup M'_k$ and then observe that $M \cap M' = \bigcup_{j,k}(M_j \cap M'_k)$; but now $M_j \cap M'_k$ can be expressed by all the equations used in M_j resp. in M'_k and by the product of the two inequalities occurring there. Hence $M_j \cap M'_k$ is a cc P-set and therefore $M \cap M'$ is a cc PA-set. For the complement $M^{\neg}$ of a cc P-set $M = \{g \neq 0,\ g_j = 0,\ j = 1, \ldots, m\}$ we see that $M^{\neg} = \bigcup_{j=1}^{m}\{g_1 = 0, \ldots, g_{j-1} = 0, g_j \neq 0\} \cup \{g_1 = 0, \ldots, g_m = 0, g = 0\}$ is a cc PA-set (the absence of an inequality can be avoided by choosing $g = 1$). Actually we have obtained a union of disjoint P-sets which is an example for the following definition. Disjoint P-sets M_j $(1 \leq j \leq m)$ with $M = \bigcup_{j=1}^{m} M_j$ form a *P-partition* of M; if an explicit procedure is given for the computation of such M_j, we say that the P-partition is *certainly computable* (cc). The intersection of two (cc) P-partitions is a (cc) P-partition again which is obtained by forming the partition consisting of all intersections of a set belonging to the first partition with a set belonging to the second one. This procedure shows that the complement of a (cc) P-partition $(\bigcup_{j=1}^{m} M_j)^{\neg} = \bigcap_{j=1}^{m} M_j^{\neg}$ yields a (cc) P-partition as each member of the intersection is a (cc) P-partition. Consequently the difference of two (cc) P-partitions is a (cc) P-partition as well. This enables us to show that (cc) P-partitions for M and M' lead to a (cc) P-partition for $M \cup M'$ by using the (cc) P-partitions for M and $M \neg M'$. This idea can be used inductively and provides an effective procedure which determines a (cc) P-partition for $\bigcup_{j=1}^{m} M_j$ if each M_j possesses a (cc) P-partition. This applies in particular if each M_j is a (cc) P-set and we see that *every* (*cc*) *PA-set possesses a* (*cc*) *P-partition*. Therefore we learn that $M^{\neg}$ and $M' \neg M$ are (cc) PA-sets provided that M and M' are (cc) PA-sets, since the statement is true for P-partitions.

DEFINITION. A *multi-valued function* $f\colon \mathbb{C}^n - \to \mathbb{C}^m$ associates with every $x \in \mathbb{C}^n$ a finite set $f(x) \subseteq \mathbb{C}^m$ (which may be empty). We introduce the *image* of a set $A \subseteq \mathbb{C}^n$ as $f(A) = \bigcup_{x \in A} f(x)$ and the *preimage* of a set $B \subseteq \mathbb{C}^m$ as $f^{-1}(B) = \{x \in \mathbb{C}^n\colon f(x) \cap B \neq \phi\}$ and define the *domain* of f as $f^{-1}(\mathbb{C}^m)$. Furthermore, we say that $F = \{(x, y)\colon y \in f(x)\} \subseteq \mathbb{C}^n \times \mathbb{C}^m$ is the *graph* of f. If $f^{-1}(\{y\})$ is finite for all $y \in \mathbb{C}^m$ it defines the *inverse function* f^{-1} and its graph is exactly F^{-1}. In case that F is a (cc) PA-relation our f is called a (*cc*) *PA-function*.

A special case of multi-valued functions are the *single-valued* functions, those for which $f(x)$ contains at most one element $(\forall x)$. Such a function is called a (*cc*) *PR-function* (piecewise rational) if its domain can be partitioned (by a given effective procedure which also computes m and n) into finitely many (cc) P-sets on each of which every coordinate of the function is a quotient of two polynomials with integral coefficients (for whose computation an effective procedure is given). If, in addition, the coordinate functions can be represented as polynomials (resp. linear functions, resp. constants) with integral coefficients, then we say that the function is *piecewise polynomial* (resp. *piecewise linear*, resp. *piecewise constant*); and we use analogous definitions for *certainly computable* (cc).

It should be noted that the domains of our functions are not necessarily all of $\mathbb{C}^n$ and that in the P-partition of the domain empty sets may occur on which the representation is arbitrary.

The above definition shows that *every* (*cc*) *PR-function is a single-valued* (*cc*) *PA-function*. That *the converse is also true* will be shown later on (Remark 5).

We want to emphasize the importance of the bijective correspondence between (PA-) functions and their graphs which are exactly the (PA-)relations with finitely many values of y for every x. This enables us to define and discuss the properties of (PA-) functions by using these graphs. For example, we make the important observation that *for a* (*cc*) *PA-function* f *the inverse* f^{-1} *is again a* (*cc*) *PA-function whenever it is defined.*

3. Basic Examples

In this section we will show that for polynomials the division with remainder and the computation of the greatest common divisor (gcd) are examples of cc PR-functions. For that purpose we will explain how finitely many rational formulae can be effectively determined which compute the coefficients of the quotient and the remainder (resp. the gcd) from the coefficients of two polynomials. The calculation of the gcd will turn out to be basic for our further discussions. Furthermore, we will explain in which way the zeros of polynomials lead to cc PA-functions.

Lemma 1 (Division with remainder). *Let* $p(z) = \Sigma_{k=0}^{s} p_k z^k$, $q(z) = \Sigma_{k=0}^{t} q_k z^k \not\equiv 0$, $d(z) = \Sigma_{k=0}^{s} d_k z^k$, $r(z) = \Sigma_{k=0}^{t} r_k z^k$ *be polynomials in* $\mathbb{C}[z]$ *with* $s, t \in \mathbb{N}_0$, *which satisfy the conditions* $p = dq + r$ *and* $\deg(r) < \deg(q)$. *This defines a relation between* $x = (p_0, \ldots p_s, q_0, \ldots, q_t)$ *and* $y = (d_0, \ldots, d_s, r_0, \ldots, r_t)$ *which is a cc PR-function* ($x - \rightarrow y$).

Proof. First we form disjoint subsets of $\mathbb{C}^{s+t+2}$ which consist of exactly all vectors x for which p and q have fixed degrees $\deg(p) = \sigma$ ($\sigma = -\infty$ or $0 \leq \sigma \leq s$), $\deg(q) = \tau$ ($0 \leq \tau \leq t$). Such a set is given as $\{x \in \mathbb{C}^{s+t+2}: p_\sigma q_\tau \neq 0,\ p_k = 0 \text{ for } \sigma < k \leq s,\ q_k = 0 \text{ for } \tau < k \leq t\}$ if $\sigma \geq 0$ resp. $\{x \in \mathbb{C}^{s+t+2}: p_k = 0 \text{ for } 0 \leq k \leq s,\ q_\tau \neq 0,\ q_k = 0 \text{ for } \tau < k \leq t\}$ if $\sigma = -\infty$. On each of these cc P-sets we give uniform rational formulae for the coefficients of d and r.

(i) If $\deg(p) < \deg(q)$ then $d = 0$ and $r = p$.

(ii) If $\deg(p) \geq \deg(q)$ then we define $\hat{q}(z) = q(z)/q_\tau z^\tau - 1$, a polynomial in $1/z$, and observe that $d(z)$ is the polynomial part of the (formal) Laurent series $(p(z)/q_\tau z^\tau)\Sigma_{k=0}^{\infty}(-\hat{q}(z))^k$ which is one interpretation of $p/q = p/q_\tau z^\tau(1 + \hat{q})$. Since terms with $k > \sigma - \tau$ do not contribute to the polynomial part it suffices to compute the coefficients of the non-negative powers of z in $(p(z)/q_\tau z^\tau)\Sigma_{k=0}^{\sigma-\tau}(-\hat{q})^k$ in order to find the coefficients of $d(z)$. Of course, this is an effective procedure and shows the required rational nature of the occurring functions. Actually they are polynomials with integral coefficients divided by a power of q_τ. The coefficients d_k with $\sigma - \tau < k \leq s$ are 0. Finally, the formula $r = p - dq$ yields an effective procedure to compute the coefficients of $r(z)$ which also have the claimed rational form. Q.E.D.

LEMMA 2 (Greatest common divisor). *Let $p(z) = \Sigma_{k=0}^{s} p_k z^k$, $q(z) = \Sigma_{k=0}^{t} q_k z^k$, $r(z) = \Sigma_{k=0}^{t'} r_k z^k$ be polynomials in $\mathbb{C}[z]$ with $s, t \in \mathbb{N}_0$ and $t' = \max(s, t)$. Assume that p and q are not both identically zero and that r is the monic (i.e., leading coefficient one) gcd of p and q. This defines a relation between $x = (p_0, \dots, p_s, q_0, \dots, q_t)$ and $y = (r_0, \dots, r_{t'})$ which is a cc PR-function ($x - \to y$).*

Proof. We apply Lemma 1 to Euclid's algorithm. For that purpose we use again $\sigma = \deg(p)$ and $\tau = \deg(q)$ to partition the x—set $\mathbb{C}^{s+t+2} \neg \{0\}$ into the cc P-sets on which σ and τ are constant. In addition to the sets described in the proof of Lemma 1 we must deal with the cases where $\tau = -\infty$ which lead to the sets $\{x \in \mathbb{C}^{s+t+2}\colon p_\sigma \neq 0,\ p_k = 0 \text{ for } \sigma < k \leq s,\ q_k = 0 \text{ for } 0 \leq k \leq t\}$. We consider each of these sets separately. If $p = 0$ (resp. $q = 0$) we are immediately done by observing that $r(z) = q(z)/q_\tau$ (resp. $r(z) = p(z)/p_\sigma$). Otherwise, we proceed as in the proof of Lemma 1 to compute the remainder $\tilde{r}(z) = \Sigma_{k=0}^{t} \tilde{r}_k z^k$. Using the notation $\rho = \deg(\tilde{r})$, we subdivide the cc P-set under consideration into the parts with fixed ρ. The case $\rho = -\infty$ is characterized by the equations $\tilde{r}_k = 0$ for $0 \leq k \leq t$ and here the gcd is $r(z) = q(z)/q_\tau$. Otherwise, we have $\tilde{r}_\rho \neq 0$ and $\tilde{r}_k = 0$ for $\rho < k \leq t$. Since the coefficients $\tilde{r}_k$ belong to $\mathbb{Z}(x)$, the equation $\tilde{r}_k = 0$ (resp. the inequality $\tilde{r}_\rho \neq 0$) is equivalent to the vanishing (resp. non-vanishing) of the numerator which belongs to $\mathbb{Z}[x]$; hence we still deal with finitely many cc P-sets. On each of these we repeat the same procedure with q and $\tilde{r}$ instead of p and q to obtain a new remainder $\hat{r}$ whose coefficients are cc rational functions of the coefficients of q and $\tilde{r}$ over $\mathbb{Z}$. Thus we can insert the formulae for the $\tilde{r}_k$ and find that the coefficients of $\hat{r}$ belong to $\mathbb{Z}(x)$ and are cc again. Since the degrees of the remainders decrease we find the gcd after at most $t + 1$ applications of this procedure. In each step we obtain cc P-sets on which an effective procedure for computing the formulae in $\mathbb{Z}(x)$ for the coefficients of the various remainders is given. This proves the claim. Note that $\deg(r)$ cannot exceed $\max(\deg(p), \deg(q))$ but can assume this maximal value if one polynomial vanishes identically. Q.E.D.

LEMMA 3 (Zeros of polynomials). *For given $n \in \mathbb{N}$ we associate with every vector $(a_0, \ldots, a_{n-1}) \in \mathbb{C}^n$ the set of solutions $x \in \mathbb{C}$ of the equation $x^n + \sum_{k=0}^{n-1} a_k x^k = 0$. This is a cc PA-function ($\mathbb{C}^n - \rightarrow \mathbb{C}$). If we associate with $(a_0, \ldots, a_{n-1})$ all complete (i.e., with multiplicities) systems of solutions $(x_1, \ldots, x_n) \in \mathbb{C}^n$ we again obtain a cc PA-function ($\mathbb{C}^n - \rightarrow \mathbb{C}^n$).*

Proof. Since a polynomial of degree n has at most n zeros we deal with a multi-valued function in the first case. Its graph is given as $\{(a_0, \ldots, a_{n-1}, x): x^n + \sum_{k=0}^{n-1} a_k x^k = 0\}$ which proves this part.

In the second case we face again only finitely many possibilities for fixed coefficients, namely, the permutations of $x_1, \ldots, x_n$. In this case the terms $a_k(-1)^{n-k}$ are exactly the elementary symmetric functions of $x_1, \ldots, x_n$ [15, p. 99]. This yields the formulae which show that we deal with a cc PA-function. Q.E.D.

4. THE MAIN LEMMA

This section is concerned with a polynomial system of equations and an inequality in one variable, where the coefficients are allowed to vary as well. For each choice of the coefficients it turns out that the whole system is equivalent to a single equation or a single inequality which can be computed from the coefficients of the system. The point is that there are finitely many situations, which correspond to cc P-sets in the coefficients, and in each of these situations the required computation corresponds to a cc rational formula. Since this naturally involves a partition it is no complication if we refine this partition by requiring that the final equation has a fixed non-negative degree and simple zeros. This will simplify the later discussions and leads to the following definition.

DEFINITION. A polynomial $p \in \mathbb{Q}[a_0, \ldots, a_n, z]$, $n \in \mathbb{N}_0$, is called *normal in z with respect to $M \subseteq \mathbb{C}^{n+1}$* if for all $(a_0, \ldots, a_n) \in M$ the following holds: The degree of p in z is always the same non-negative number, and the discriminant (gcd of p and p') has no zeros.

MAIN LEMMA. *Let $g_j(z) = \sum_{k=0}^{n} a_{jk} z^k$ $(j = 1, \ldots, m)$ and $g(z) = \sum_{k=0}^{n} b_k z^k$ for $m, n \in \mathbb{N}$ with $(a, b, z) = (a_{10}, \ldots, a_{mn}, b_0, \ldots, b_n, z) \in$*

$\mathbb{C}^{(m+1)(n+1)+1}$. *Then the vectors* (a, b, z) *satisfying*

$$g(z) \neq 0, \qquad g_j(z) = 0 \qquad (j = 1, \ldots, m) \qquad (*)$$

form a cc P-set. The space $\mathbb{C}^{(m+1)(n+1)}$ *of all possible coefficients* (a, b) *can be decomposed (disjointly) into the set* $\{a = 0\}$, *where* $(*)$ *is equivalent to* $g(z) \neq 0$, *and into finitely many P-sets M on each of which* $(*)$ *is equivalent to* $h(z) = 0$ *with a suitable polynomial* $h \in \mathbb{Z}[a, b, z]$ *which is normal in z with respect to this M. Furthermore, an effective procedure depending only on* (m, n) *is given for the computation of a possible system of P-sets M and possible polynomials h.*

Proof. The non-trivial part of the claim is the decomposition of the set $\{a \neq 0, b \text{ arbitrary}\}$ with the prescribed properties. To prove it we decompose the set $\{a \neq 0\}$ into the cc PA-sets M_j $(1 \leq j \leq m)$ defined by the requirement that $g_1, \ldots, g_{j-1}$ vanish identically while $g_j(z) \not\equiv 0$ for the coefficient vectors in M_j. On M_j we compute successively the monic gcd r_l of r_{l-1} and g_{l+1} for $j \leq l \leq m$ (put $r_{j-1} = g_j$, $g_{m+1} = 0$) by applying Lemma 2. In each of these (at most m) steps we obtain by an effective procedure a successively refined decomposition of M_j into finitely many PA-sets on each of which every coefficient of r_l is given by a cc rational formula over $\mathbb{Z}$ in the coefficients of r_{l-1} and g_{l+1} and hence (by insertion) in terms of the original coefficients a; this argument will also be used in each of the following steps. Note that r_m represents the monic gcd of $g_1, \ldots, g_m$ on each M_j, and we have shown so far that its coefficients are cc PR-functions with domain $\{a \neq 0\}$. Next we compute the derivative $r_m'(z)$ and then $d(z)$ as the monic gcd of r_m and r_m' and again the coefficients are cc PR-functions. Using Lemma 1 we compute (cc) $\tilde{r}_m = r_m(z)/d(z)$ which is a monic polynomial without multiple zeros in z for every choice of $a \neq 0$. Then we calculate (cc) the monic gcd $\tilde{d}(z)$ of $\tilde{r}_m(z)$ and $g(z)$ as well as $\tilde{r}(z) = \tilde{r}_m(z)/\tilde{d}(z)$. We see that $\tilde{r}(z)$ is a monic polynomial with simple zeros whose coefficients are cc PR-functions with domain $\{a \neq 0, b \text{ arbitrary}\}$. Note that $(*)$ is equivalent to $\tilde{r}_m(z) = 0$ and $g(z) \neq 0$ while $\tilde{r}_m(z) = 0 = g(z)$ means $\tilde{d}(z) = 0$. Hence $(*)$ is equivalent to $\tilde{r}_m(z) = 0$ and $\tilde{d}(z) \neq 0$, which in turn is equivalent to $\tilde{r}(z) = 0$, since $\tilde{r}_m$ has simple zeros. Finally we explain how to calculate (cc) h from $\tilde{r}$. By refining our decomposition into P-sets we can arrange that $\tilde{r}$ has on each of these P-sets a fixed representation in $\mathbb{Z}(a, b)[z]$ with fixed degree. Here the coefficients are given as quotients of cc polynomials in $\mathbb{Z}[a, b]$, where all the denominators are different from zero on the P-set under consideration. By multiplying $\tilde{r}(z)$ with all these denominators we obtain $h(z)$ having all the properties claimed. Q.E.D.

Remark 2. The Main Lemma remains true if the involved polynomials have different degrees or if no equations occur at all. These cases can be incorporated by choosing certain parameters to be zero which directly leads to additional conditions for the occurring P-sets.

5. Inductive Resolution

The inductive application of the Main Lemma will now enable us to decompose an arbitrary PA-relation into P-sets in (x, y) of a particularly simple structure. This decomposition will play a central role in our later discussions.

Definition. A relation M is called *normal* if a permutation of the original y-variables exists (final notation: $y_1, \dots, y_m$), an integer r ($0 \leq r \leq m$) and a P-set $\tilde{M}$ in x such that M is a P-set in (x, y) given by the conditions $x \in \tilde{M}$ and $g(x, y_1, \dots, y_r) \neq 0$, $g_j(x, y_1, \dots, y_{r+j}) = 0$ for $1 \leq j \leq m - r$. Here g and g_j belong to $\mathbb{Z}[x, y]$, g does not vanish identically for each $x \in \tilde{M}$, and for every j the polynomial g_j is normal and non-constant in y_{r+j} with respect to the set M_j of vectors $(x, y_1, \dots, y_{r+j-1})$ satisfying $x \in \tilde{M}$ and $g(x, y_1, \dots, y_r) \neq 0$, $g_k(x, y_1, \dots, y_{r+k}) = 0$ for $1 \leq k < j$. Given such a representation for M we call $y_1, \dots, y_r$ the *independent* and $y_{r+1}, \dots, y_m$ the *dependent* variables. We say that such a normal representation for M is cc if we give an effective procedure which computes the integers m, n, r, the permutation of the y-variables, the P-set $\tilde{M}$, and all the polynomials g, g_j.

We like to emphasize the simple inductive structure of the normal relation M: the possible values of x are given by $x \in \tilde{M}$; after fixing x the possible values of $(y_1, \dots, y_r)$ are given by $g(x, y_1, \dots, y_r) \neq 0$, and the possible values of $(x, y_1, \dots, y_r)$ lie in M_1; after fixing these the possible values of y_{r+1} are given by $g_1(x, y_1, \dots, y_r, y_{r+1}) = 0$, thereby defining M_2; in each further step the possible values of a new dependent variable y_{r+j} can be calculated from previously computed values by *solving* a single polynomial equation which has a fixed positive number of simple zeros; the process terminates with $j = m - r$ at which time all possible values of the dependent variables are calculated in terms of $x, y_1, \dots, y_r$ (inductive resolution).

Theorem 1 (Decomposition Theorem). *Every PA-relation can be decomposed into finitely many, disjoint normal relations. If the given PA-relation is cc an effective procedure is given which calculates representations of all normal relations belonging to a possible decomposition.*

Proof. By Remark 1 it suffices to prove the theorem for a P-set $\{(x, y)$: $g(x, y) \neq 0$, $g_j(x, y) = 0$ for $1 \leq j \leq m'\}$. For that purpose we apply the Main Lemma with $z = y_m$. This yields a decomposition of all

$(x, y_1, \ldots, y_{m-1}) = (x, \hat{y})$ into finitely many, disjoint P-sets such that on one of them the original conditions $g \neq 0$, $g_j = 0$ are equivalent to $g \neq 0$ while on each of the others they are equivalent to $h(x, y) = 0$ with a suitable polynomial $h \in \mathbb{Z}[x, y]$ which is normal in y_m with respect to the P-set under consideration. If h has positive degree then it yields the desired polynomial equation for y_m and we continue by working on the underlying P-set in $(x, \hat{y})$ which has fewer y-variables. If the degree of h is zero then $h = 0$ is impossible on the underlying P-set so that this part leads to an empty (x, y)-set which can be disregarded. The remaining case corresponds to $a = 0$ in the Main Lemma which means that all coefficients of the g_j as polynomials in y_m are required to vanish. These are polynomial equations in $\mathbb{Z}[x, \hat{y}]$ while the inequality $g \neq 0$ remains untouched. In this situation we apply the Main Lemma again with respect to the variable $z = y_{m-1}$ and arrive as before at a normal equation with positive degree for y_{m-1} with respect to a P-set in all other variables or we are left with the same inequality $g \neq 0$ and the conditions that all coefficients in our present equations for y_{m-1} are zero. These are polynomial equations which contain neither y_{m-1} nor y_m, to which the process can be applied again. Hence after at most m applications of our Main Lemma we arrive at the following *equivalent description of the original P-set in terms of disjoint parts: For each k, $1 \leq k \leq m$, we obtain finitely many P-sets in $(x, \hat{y}_k)$, where $\hat{y}_k$ is y without y_k, and on these P-sets the original system $g \neq 0$, $g_j = 0$ is equivalent to an equation $h_k(x, y) = 0$, where $h_k \in \mathbb{Z}[x, y]$ has positive degree and is normal in y_k with respect to the underlying P-set. In addition, we obtain one P-set in x on which the original system $g \neq 0$, $g_j = 0$ is equivalent to* $g(x, y) \neq 0$. This P-set can be restricted to those x for which $g(x, y)$ is not identically zero in y and can, therefore, be decomposed further (cf. Remark 1) into P-sets in x, where this additional condition is satisfied. These P-sets together with the condition $g(x, y) \neq 0$ define normal relations. On the P-sets in $(x, \hat{y}_k)$, however, we may proceed inductively in order to obtain the claimed decomposition. Since this induction terminates after at most m steps we obtain an effective procedure that calculates representations of all normal relations belonging to a possible decomposition by using the effective procedures given in Remark 1 and our Main Lemma. Q.E.D.

The Decomposition Theorem can be interpreted as the solution of a system of equations in the unknowns y subject to an inequality where everything depends on the parameters x. In contrast to other authors [11, pp. 269–271; 14, pp. 59–65] we are not content to obtain a procedure for the computation of the solution but we also want to reveal the structure of the set of solutions and discuss the dependence upon the parameters.

Every normal relation F' (with cc representation) which is a subset of the given (cc) PA-relation F can occur in such a decomposition, since we need only apply the Decomposition Theorem to the (cc) PA-relation $F \neg F'$.

If we drop x and $\tilde{M}$ in the definition of a normal relation we obtain the definition of a *normal* set M; in the case $r = 0$ the polynomial $g_1 \in \mathbb{Z}[y_1]$ should have positive degree and only simple zeros (such a normal set consists of finitely many points); if $r \geq 1$ we require $g(y_1, \ldots, y_r) \not\equiv 0$ so that M is always non-empty.

Remark 3. It is clear that the Decomposition Theorem holds in an analogous manner for PA-sets instead of PA-relations including the cc-version. Then the above proof also yields an effective procedure to check whether a cc PA-set is empty. For that purpose we disregard all P-sets with $g \equiv 0$ which are obviously empty and determine a decomposition into normal sets for the rest. If in this process one of the polynomials h as constructed above has degree zero, then the corresponding P-set is empty since $h = 0$ is a contradiction. Otherwise we find a non-empty normal subset of the PA-set under consideration.

6. Cardinality of Sections

Now want to discuss how many values for the dependent variables are possible if the independent ones are fixed. Here the normal P-sets play an important role because for them we obtain a fixed, finite number. These results yield a global version of the implicit function theorem as an immediate consequence.

Remark 4. Let us consider a normal relation M with a fixed normal representation and a fixed vector $x \in \tilde{M}$ such that $g(x, y_1, \ldots, y_r) \not\equiv 0$. The following arguments remain valid if M is a normal set instead in which case x is absent. For given values of the independent variables $y_1, \ldots, y_{r-1}$ such that g does not vanish identically we know that y_r can assume any complex value with the exception of the finitely many values for which g becomes zero. The same is true for any other of the independent variables. If we fix the values of $y_1, \ldots, y_r$ such that $g(x, y_1, \ldots, y_r) \neq 0$ then we can find exactly d different vectors of values for the dependent variables $y_{r+1}, \ldots, y_m$, where d is the product of the degrees of the polynomials g_j. Thus we see that (for fixed x) M defines a multi-valued function $((y_1, \ldots, y_r) - \rightarrow (y_{r+1}, \ldots, y_m))$ unless there are no independent or no dependent variables (if M is a normal set we actually deal with a PA-function). For such a function the vector of dependent variables can locally be represented by d different, single-valued, analytic functions in the independent variables on the set $\{g \neq 0\}$. Each of these functions is uniquely determined by its value at a single fixed vector $(y_1, \ldots, y_r)$ belonging to the open set where this representation holds since $\{g \neq 0\}$ is an open, connected set in $\mathbb{C}^r$. Moreover, we can continue these functions analytically as long as we stay in the set $\{g \neq 0\}$. For $r = 1$ and $m = 2$ this leads to function elements in the sense of Ahlfors [1, p. 292]. His results applied to each dependent variable successively yield the above statements if we recall that we deal with normal polynomials g_j in the representation of M.

THEOREM 2 (Cardinality Theorem). *Let a (cc) PA-relation F in (x, y) be given. If we associate with every $x \in \mathbb{C}^n$ the cardinality of the set $\{y \in \mathbb{C}^m: (x, y) \in F\}$ (for convenience taken to be -1 if the set is not finite) then we obtain a (cc) piecewise constant function.*

Proof. First we decompose F into normal relations F_i $(1 \leq i \leq N)$. For each of those we have a P-set $\tilde{M}_i$ in x and know the following. If there exists at least one independent variable then, for every $x \in \tilde{M}_i$, we have infinitely many "solutions" y with $(x, y) \in F_i$, i.e., the cardinality function for F_i is -1 on $\tilde{M}_i$. Otherwise, we deal only with dependent variables and hence find the same fixed number of solutions for every $x \in \tilde{M}_i$ according to Remark 4; this gives the value of the cardinality for F_i on $\tilde{M}_i$. Of course, the cardinality for F_i is 0 on $\tilde{M}_i^{\neg}$. Now consider all the possible intersections $\bigcap_{i=1}^{N} S_i$, where S_i is either $\tilde{M}_i$ or $\tilde{M}_i^{\neg}$; they form a decomposition of $\mathbb{C}^n$. On such an intersection the desired cardinality for F can be found to be -1 if the cardinality for at least one F_i is -1 on S_i. Otherwise, the cardinality for F is just the sum of the cardinalities for the F_i on S_i.

Since each intersection can be decomposed into P-sets (Remark 1) we obtain a P-partition of $\mathbb{C}^n$ by an effective procedure which also computes the desired cardinality as a fixed integer on each occurring P-set. Q.E.D.

Remark 5. If the cardinality only assumes the values 0 or 1 then F actually defines a PR-function. This can be seen from a normal decomposition of F as in the proof above. Since at most one solution y occurs for every x we deal only with dependent variables where each equation g_j is linear in y_j on $\tilde{M}_i$ and the $\tilde{M}_i$ must be disjoint. Hence y_j is on $\tilde{M}_i$ a rational function in $\mathbb{Z}(x, y_1, \ldots, y_{j-1})$ and by insertion a rational function in $\mathbb{Z}(x)$. This effective procedure shows that the PR-function is cc if the original PA-function is cc.

It is not surprising that the solution of a system of equations is related to the implicit function theorem. Yet it is interesting to notice that a global version of this theorem can be deduced from the Decomposition Theorem.

THEOREM 3. *Let* $x \in \mathbb{C}^n$, $y \in \mathbb{C}^m$, *and* $g_j \in \mathbb{Z}[x, y]$ *for* $1 \leq j \leq m$ $(m, n \in \mathbb{N})$. *Then the set* $F = \{g_j(x, y) = 0$ *for* $1 \leq j \leq m,\ g(x, y) = \det[\partial g_j / \partial y_k] \neq 0\}$ *defines a PA-function* $(x - \to y)$.

Proof. Of course, F is a PA-relation in (x, y) and we need to show that for every fixed x there are only finitely many solutions y with $(x, y) \in F$. This is done in two steps:

(i) Given such a y the local implicit function theorem [3, pp. 267 ff] tells us that in a suitable neighborhood of y there are no further solutions with the same x.

(ii) We consider a possible normal decomposition of the PA-relation F according to the Main Theorem. If in any of the occurring normal relations ($\neq \varnothing$) there were independent variables then we could apply the observations of Remark 4: Either we have dependent variables that yield analytic functions so that the y-values for our fixed x are not isolated; or all y-variables are independent with the consequence that in the normal part of F under consideration the y-variables are only restricted by the condition $g(x, y) \neq 0$ so that the y-values for our fixed x are again not isolated. Since both cases contradict (i) we have only dependent variables on the normal parts and thus only finitely many solution vectors y. Q.E.D.

It should be noted that on any normal set we can name the independent variables as x and the dependent ones as y; then $[\partial g_j / \partial y_k]$ is a triangular matrix whose determinant is not zero on the set due to the conditions imposed on the g_j.

7. THE DIMENSION OF A PA-SET

It is also possible to introduce the notion of dimension, which is especially important for systems of linear equations, for our polynomial situation and use it to decide whether a PA-set admits a further inductive resolution according to the Decomposition Theorem. We will interpret the dimension algebraically as well as topologically or in the sense of measure theory.

DEFINITION. The transcendency degree over $\mathbb{Q}$ of a vector $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ is defined as the transcendency degree of the set $\{x_1, \ldots, x_n\}$. The algebraic *dimension* of a non-empty set $M \subseteq \mathbb{C}^n$ is the maximal transcendency degree of its elements. The dimension of the empty set is defined to be -1.

For a normal set ($\neq \varnothing$) the dimension is immediately identified as the number r of independent variables. Therefore we can calculate the dimension of a PA-set by using a normal decomposition and locating the part of highest dimension. This can be done by an effective procedure if the PA-set is cc.

Remark 6. The relation between the dimension and normal decompositions enables us to decide whether a P-set $\{x \in \mathbb{C}^n\colon g(x) \neq 0, g_j(x) = 0\}$ has dimension n. This happens if this P-set contains a normal set without dependent variables. When we follow the proof of the Decomposition Theorem we see that in each step the inequality $g(x) \neq 0$ is reproduced. Hence such a normal set can only have the form $\{x \in \mathbb{C}^n\colon g(x) \neq 0\}$. From that we deduce that in the original P-set we have $g_j(x) \equiv 0$ $(\forall j)$. This proves that *a P-set in* $\mathbb{C}^n$ *has dimension n if and only if all of its defining equations are trivial and in the inequality* $g(x)$ *is not identically zero*. We call such a set a *free* P-set. The intersection of two free P-sets is again a free P-set (in particular non-empty) defined by the product of the two inequalities. Therefore, *a PA-set in* $\mathbb{C}^n$ *has dimension n if and only if it is the disjoint union of a single free P-set and a PA-set of dimension at most* $n - 1$. Therefore a PA-set in $\mathbb{C}^n$ has dimension at most $n - 1$ if and only if its complement has dimension n (hence it contains a free P-set defined by the inequality $g \neq 0$). This leads to the conclusion that *a PA-set in* $\mathbb{C}^n$ *has dimension at most* $n - 1$ *if and only if there exists a polynomial g, which does not vanish identically on* $\mathbb{C}^n$, *such that* $g(x) = 0$ *for all x in the PA-set.*

So far we have used an algebraic concept of dimension and we will now show that this agrees with the measure-theoretic or the topological concept.

DEFINITION. For a set $S \subseteq \mathbb{R}^N$ ($N \in \mathbb{N}$ fixed) the *p-dimensional outer Hausdorff-measure* ($p \geq 0$, $p \in \mathbb{R}$) is defined as $H_p(S) = \sup_{\varepsilon > 0} \inf \sum_{j=1}^{\infty} d^p(A_j)$, where $S \subseteq \bigcup_{j=1}^{\infty} A_j$ and the diameters $d(A_j) < \varepsilon$ for all j [6, pp. 102/103]. Moreover, S is called *σ_p-finite* if $S = \bigcup_{j=1}^{\infty} S_j$ with $H_p(S_j) < \infty$ for all j; if in addition $H_p(S) > 0$, we say that S has *precise dimension p*. More generally one defines the *Hausdorff-dimension* of S as the infimum over all $p \geq 0$ for which $H_p(S) = 0$. We use the real Hausdorff-dimension for our complex sets by splitting each complex variable into its real and imaginary parts.

Furthermore, the set S (as a topological space) has topological dimension at most r ($r \in \mathbb{Z}$, $r \geq -1$) if at each of its elements there exist arbitrarily small neighborhoods whose boundaries have topological dimension at most $r - 1$, where the empty set has dimension -1. The least possible among these numbers r is the *topological dimension* of S [6, p. 21].

PROPOSITION 1. *Let M be a non-empty PA-set with algebraic dimension r. Then M has precise as well as topological dimension* $2r$.

Proof. Let us first assume that M is a normal set with independent variables $y_1, \ldots, y_r$ and dependent variables $y_{r+1}, \ldots, y_n$. For $r = 0$ (finitely many points according to the definition of a normal set) and $r = n$ (whole space excluding the points with $g(y_1, \ldots, y_n) = 0$) the assertion is trivial [6, p. 44]. Hence we may assume $1 \leq r < n$.

Due to Remark 4 we know that M is locally homeomorphic to $\mathbb{C}^r$ via the projection onto $y_1, \ldots, y_r$. If we use $y_1, \ldots, y_r$ locally as complex coordinates we see that M *is*, in fact, *a complex manifold which is analytically embedded into* $\mathbb{C}^n$. Since a fixed, finite number of points has coordinate vector $(y_1, \ldots, y_r)$ and since the coordinate vectors form an open set in $\mathbb{C}^r$ which can be covered by countably many small compact sets, it is clear that the topological dimension of M is exactly $2r$ [6, p. 30] and that M is an F_σ-set.

From Remark 4 we also learn that the dependent variables can locally be represented by d analytic functions which are in particular locally Lipschitz. Hence we consider a non-empty open bounded set $U \subseteq \mathbb{C}^r$ and a function $f: U \to \mathbb{C}^{n-r}$ that is Lipschitz on U with constant L. Then we take $V = \{(y, f(y)): y \in U\} \subseteq \mathbb{C}^n$ and observe that $H_p(U) \leq H_p(V) \leq (1 + L^2)^{p/2} H_p(U)$ holds due to corresponding inequalities for diameters. Therefore V has the same precise dimension as U, namely $2r$, and by σ-additivity this is also the precise dimension of M.

If M is an arbitrary PA-set we decompose it into finitely many normal sets to which we can apply the preceding arguments. Since the local dimension is the maximum of the single dimensions in each case, our result follows [6, p. 30]. Q.E.D.

It is also worthwhile to notice that PA-functions preserve dimensional properties of arbitrary sets.

PROPOSITION 2. *Let f be a PA-function ($\mathbb{C}^n - \to \mathbb{C}^m$) and A be an arbitrary set in $\mathbb{C}^n$. Then the following hold* ($p \geq 0$, $p \in \mathbb{R}$):

(i) *The algebraic dimension of $f(A)$ is not greater than the algebraic dimension of A,*

(ii) $H_p(A) = 0 \Rightarrow H_p(f(A)) = 0$,

(iii) *A is σ_p-finite $\Rightarrow f(A)$ is σ_p-finite.*

Proof. The first assertion is obvious since the elements of $f(x)$ depend algebraically on those of x [15, p. 225]. For (ii) and (iii) we decompose the graph of f into normal relations where x is restricted to a P-set while all the y-variables are dependent. The equations for the y-variables can also be solved analytically if we vary x freely in a small neighborhood. Hence f is locally given by d Lipschitz-functions, and the effect of the total PA-function can be described by countably many such pieces. Since

Lipschitz-functions preserve zero measure and σ-finiteness the result follows. Q.E.D.

Proposition 2 shows in particular that a PA-function increases neither the Hausdorff dimension nor the algebraic dimension.

8. Properties of PA-functions

Our next goal is to discuss the properties of PA-functions and PA-sets. Instead of an algebraic point of view we consider our functions as multi-valued mappings. Therefore we deal with sets and mappings and want to show that our system possesses remarkable closure properties with respect to many finite operations. All of these properties can be deduced from four basic ones which are stated below together with those given in Remark 1.

LEMMA 4. (i) *The projection from* $\mathbb{C}^n$ *onto* $\mathbb{C}^{n-1}((x_1,\dots,x_n)\to(x_1,\dots,x_{n-1}))$ *maps (cc) PA-sets onto (cc) PA-sets* ($n\geq 2$).

(ii) *The preimage of a (cc) PA-set in* $\mathbb{C}^{n-1}$ *under the above projection is a (cc) PA-set in* $\mathbb{C}^n$ ($n\geq 2$).

(iii) *A permutation of the variables (for whose computation an explicit procedure is given) maps (cc) PA-sets onto (cc) PA-sets.*

(iv) *Sum, difference, product, and quotient are cc PA-functions* ($\mathbb{C}^2 - \to \mathbb{C}$).

Proof. (i) We put $\tilde{x}=(x_1,\dots,x_{n-1})$, $y=x_n$ and interpret the (cc) PA-set $M\subseteq\mathbb{C}^n$ as a (cc) PA-relation in $(\tilde{x},y)$. Then we define the cardinality function (of $\tilde{x}$) as in Theorem 2 ($\mathbb{C}^{n-1} - \to \mathbb{C}$) and observe that the projection of M consists of those $\tilde{x}\in\mathbb{C}^{n-1}$ for which the cardinality is not zero which is obviously a (cc) PA-set.

(ii) This is obvious since x_n is not restricted.

(iii) This is obvious, too.

(iv) The quotient maps $(x_1,x_2)\in\mathbb{C}^2$ with $x_2\neq 0$ onto $x_1/x_2\in\mathbb{C}$. Its graph can be written as $\{x_1-x_2y=0,\ x_2\neq 0\}\subseteq\mathbb{C}^2\times\mathbb{C}$ which is a cc PA-relation. The rest follows analogously.
The certain computability is always straightforward. Q.E.D.

Parts (i) and (ii) are also true for the general projections $pr_{nm}(x_1,\dots,x_{n+m})=(x_1,\dots,x_n)$ and $\tilde{pr}_{nm}(x_1,\dots,x_{n+m})=(x_{n+1},\dots,x_{n+m})$ from $\mathbb{C}^{n+m}$ onto $\mathbb{C}^n$ (resp. $\mathbb{C}^m$) as can be seen by induction using (i), (ii), or (iii).

DEFINITION. Let $f\colon \mathbb{C}^n - \to \mathbb{C}^m$ and $g\colon \mathbb{C}^{n'} - \to \mathbb{C}^{m'}$ be given multi-valued functions with respective graphs F, G. Then we define the following multi-valued functions: the *cross-product* $f \times g\colon \mathbb{C}^n \times \mathbb{C}^{n'} - \to \mathbb{C}^m \times \mathbb{C}^{m'}$ by $f \times g(x, x') = f(x) \times g(x')$; in case $n' = m$ the *composition* $g \circ f\colon \mathbb{C}^n - \to \mathbb{C}^{m'}$ by $g \circ f(x) = g(f(x))$; in case $n' = n$, $m' = m$ the *join* $f \vee g\colon \mathbb{C}^n - \to \mathbb{C}^m$ by $f \vee g(x) = f(x) \cup g(x)$, the *common part* $f \wedge g\colon \mathbb{C}^n - \to \mathbb{C}^m$ by $f \wedge g(x) = f(x) \cap g(x)$, and the *excision* $f \neg g\colon \mathbb{C}^n - \to \mathbb{C}^m$ by $f \neg g(x) = f(x) \neg g(x)$.

For the resulting maps it is obvious that the image of a single point is always a finite set, and it is easy to explain the graphs of these maps in terms of F and G. In the sequel we will use the notations introduced in Section 2.

THEOREM 4. *(a) Let $A \subseteq \mathbb{C}^n$, $B \subseteq \mathbb{C}^m$ be (cc) PA-sets and $f\colon \mathbb{C}^n - \to \mathbb{C}^m$, $g\colon \mathbb{C}^{n'} - \to \mathbb{C}^{m'}$ (cc) PA-functions. Then the following sets are (cc) PA-sets: $A \times B, f(A), f^{-1}(B)$; and the following are (cc) PA-functions: $f \times g$, $g \circ f$ (for $m = n'$) and $f \vee g, f \wedge g, f \neg g$ (for $n = n'$, $m = m'$).*

(b) *If $f\colon \mathbb{C}^n - \to \mathbb{C}^2$ with graph F is a (cc) PA-function then so are the multi-valued functions with graphs $\{(x, y_1 * y_2) : (x; y_1, y_2) \in F\}$ with $*$ being $+, -$, or $\cdot$ (resp. $\{(x, y_1/y_2) : (x; y_1, y_2) \in F,\ y_2 \neq 0\}$).*

(c) *A relation F is the graph of a (cc) PA-function $\mathbb{C}^n - \to \mathbb{C}^m \times \mathbb{C}^p (x - \to (y, z))$ if and only if its projection under $(x, y, z) \to (x, y)$ defines a multi-valued function $(x - \to y)$ and F can also be interpreted as the graph of a (cc) PA-function $((x, y) - \to z)$.*

Proof. (a) We denote by F, G the graphs of f, g. Then we have $A \times \mathbb{C}^m = pr_{nm}^{-1}(A)$ and $\mathbb{C}^n \times B = \tilde{pr}_{nm}^{-1}(B)$ which together with Remark 1 and Lemma 4 prove the claim for $A \times B = (A \times \mathbb{C}^m) \cap (\mathbb{C}^n \times B)$, $f(A) = \tilde{pr}_{nm}(F \cap (A \times \mathbb{C}^n))$, $f^{-1}(B) = pr_{nm}(F \cap (\mathbb{C}^n \times B))$. For $f \vee g$, $f \wedge g$, and $f \neg g$, it suffices to observe that their respective graphs are $F \cup G$, $F \cap G$, and $F \neg G$; moreover, the graph of $f \times g$ is obtained from $F \times G$ by a suitable permutation so that the previous argument can be used. Thus we are left with examining the graph of $g \circ f$. This graph is obtained by intersecting $F \times G$ with $\mathbb{C}^n \times \{(y, y)\colon y \in \mathbb{C}^m\} \times \mathbb{C}^{m'}$ (apply Remark 1 and (ii), (iii) of Lemma 4) and projecting the intersection onto the first n and the last m' variables by simply disregarding the $2m$ variables inbetween (apply (i) and (iii) of Lemma 4).

(b) The functions under consideration are obtained by composing f with sum, difference, product, or quotient. Hence the claim follows from part (a) and (iv) of Lemma 4.

(c) The relation F is the graph of a (cc) PA-function if and only if F is a (cc) PA-relation which associates with x only finitely many pairs (y, z), i.e., with x only finitely many y and with (x, y) only finitely many z. This can be reinterpreted as stated in the theorem. Q.E.D.

We observe that parts (a) and (b) of Theorem 4 can be generalized from PA-functions to PA-relations in a natural way.

While part (a) contains the set-theoretic operations (the inverse function was already discussed in Section 2), we present in part (b) our concept of the algebraic operations. Before we perform such an operation with two PA-functions f_1, f_2: $\mathbb{C}^n - \to \mathbb{C}$ we first discuss whether and how the two dependent variables y_1 and y_2 are algebraically related. Let us explain this with the example of functions f_j: $\mathbb{C} - \to \mathbb{C}$ $(j = 1,2)$ given by the same relations $y_j^2 - x = 0$ for $x \neq 0$. Now we can define $y_1 + y_2$ by adding all the possible values of y_1 to all those of y_2 and thereby obtain three values for every $x \neq 0$. But it also makes sense to require that we only add identical values of y_1 and y_2, since both of them are defined by the same relation; then the sum assumes only two values for every $x \neq 0$. Both possibilities are incorporated in part (b). In the first case where y_1 and y_2 are independent of each other we define f: $\mathbb{C} - \to \mathbb{C}^2$ by the two relations $y_j^2 - x = 0$ $(j = 1,2)$ for $x \neq 0$, whereas in the second case we adjoin the relation $y_1 - y_2 = 0$ which gives the dependence of the two variables. This example shows the great flexibility of this concept which seems especially suited in the presence of multi-valuedness.

Of course part (b) generalizes to vectors $y_1, y_2 \in \mathbb{C}^m$ where we apply the operations to each coordinate separately; this is proven as soon as we know that our basic property (iv) of Lemma 4 generalizes accordingly which is a consequence of the following discussion.

The final part (c) of Theorem 4 reveals how the property that f is a PA-function is related to properties of its coordinate functions. Since the y-variables may not be independent of each other it is not sufficient to consider each coordinate function separately; in addition we must clarify its interdependence with the other coordinates. However, this is not necessary if the coordinate-function is single-valued as we will explain now. For the multi-valued function f: $\mathbb{C}^n - \to \mathbb{C}^{m+p}$ which associates with x finitely many pairs (y, z) we define the coordinate functions f_y: $\mathbb{C}^n - \to \mathbb{C}^m$ and f_z: $\mathbb{C}^n - \to \mathbb{C}^p$ by appropriate projections of the graph of f. *In the case that f_y is single-valued* we will show that *f is a (cc) PA-function if and only if f_y and f_z are (cc) PA-functions.* For that purpose we observe that f associates with every fixed x in its domain vectors (y, z), all of them with the same $y \in \mathbb{C}^m$; for fixed x the points $(x; y, z)$ in the graph of f are therefore determined as $\{(x, y, z) : (x, y)$ in the graph of f_y, $z \in \mathbb{C}^p\}$ intersected with $\{(x, y, z) : (x, z)$ in the graph of f_z, $y \in \mathbb{C}^m\}$.

In particular, f is a single-valued (cc) PA-function ($\mathbb{C}^n - \to \mathbb{C}^m$) if and only if each of its coordinate functions is a single-valued (cc) PA-function

($\mathbb{C}^n - \to \mathbb{C}$). This property allows the extension of (iv) in Lemma 4 to vectors with more than one coordinate as we claimed above.

9. Piecewise Rational Programs

Before we discuss various applications of PA-functions and PA-sets we want to demonstrate their close connection with certain algorithms. This provides a powerful link between the theory which has been developed so far and many concrete situations.

A *P-program* consists of numbers $m, n, q \in \mathbb{N}$, a variable vector $x = (x_1 \dots, x_n)$ in $\mathbb{C}^n$, a fixed parameter vector $c = (c_1, \dots, c_q) \in \mathbb{C}^q$, and a finite combinatorial structure, which explains how the program can be applied to all $x \in \mathbb{C}^n$ for the selected c to calculate m numbers $y_1, \dots, y_m \in \mathbb{C}$ (if possible) by a well-defined procedure which will be explained in the following: There are consecutive steps numbered $1, 2, \dots,$ which are classified as tests, computations or the result. A *test* refers by index to an earlier computation or a coordinate of x or c. If the test is actually applied to a complex number we receive the information whether or not this number is zero. A *computation* specifies one of the four arithmetic operations ($+, -, \cdot, \div$) and two places which are earlier computations or coordinates of x or c. If the computation is actually applied to complex numbers we receive the resulting number, and the program as such must guarantee that division by zero will not occur. For each step its (combinatorial) nature is completely determined by the nature of all earlier steps including the assumed information gained at the test steps. We call this the *present information*, which results in a tree-like structure. So the nature of the first steps is clear until the first test occurs, at which moment the program continues in two separate branches depending on the assumed gain in information and so on. There are only finitely many branches (at least one), and each branch ends with the result step. It is an important requirement for each branch that all computations occurring there are possible if x is considered as a vector of indeterminants, i.e., computations in the field $\mathbb{C}(x)$. The last step of the branch, the *result*, either yields "empty" or refers by index to m places which are earlier computations or coordinates of x or c. The present information alone determines whether or not this is the result step, and if it is, whether or not the result is "empty", and if not, which m places are to be used to make up the result. This completes the description of the combinatorial structure of the program. The P-program itself also requires the selection of c and is to be applied to all $x \in \mathbb{C}^n$. If $c \in \mathbb{Z}^q$ we speak of a *PR-program*. For each x all actions of the program take place within a single branch of the combinatorial structure. At each step we consider as *known quantities* all numbers which were actually computed earlier including the coordinates of x and c. We denote by $N(x)$ the step number of the result and by $N_0(x)$ the number of computations used. Both numbers are bounded, and we call $N_0 = \sup_x N_0(x)$ the *computational complexity* of the program.

The programs defined above are special cases of the "computation tree" discussed in [13]. The main difference lies in our assumptions which guarantee that our program works formally as well as actually in all possible cases.

It is natural to represent the combinatorial structure of a P-program in the form of a flow-chart. Let us take a PR-program and follow the computational steps of a fixed branch interpreted in $\mathbb{Z}(x)$. At each step we obtain a quotient of two polynomials in $\mathbb{Z}[x]$, whose denominator is not zero in $\mathbb{Z}[x]$. In connection with a test step we only consider the numerator polynomial. The branching after a test can be indicated by a condition $= 0$ (resp. $\neq 0$), and all these conditions which occur along a fixed branch can be used in a natural way to define a P-set in x associated with that branch. From their construction it is clear that the P-sets corresponding to all branches form a P-partition of $\mathbb{C}^n$. With those branches which do not end with the result "empty" there are associated m rational functions written as quotients of polynomials in $\mathbb{Z}[x]$. We shall show that their denominators have no zero on the P-set corresponding to that branch. Therefore these branches can be used to define a PR-function ($\mathbb{C}^n - \rightarrow \mathbb{C}^m$), which is uniquely associated with our PR-program. In fact, if $x \in \mathbb{C}^n$ belongs to the P-set associated with a particular branch the application of the program will proceed exactly along this branch and all required divisions will be possible, so that the actually computed values will be values of the rational functions which we already associated with the computational steps. If the branch does not end with the result "empty", the program actually calculates the values of the m rational functions mentioned before. In particular, it follows that their denominators are not zero on the whole P-set under consideration as claimed. In summary, we see that the program calculates exactly the values of our PR-function on its domain and yields "empty" outside of it. In this situation we say that the program *calculates* the PR-function. (We note that certain P-sets may be empty so that the corresponding branches will never be used. This could also happen for branches whose result is not "empty".)

When we say that the combinatorial structure of a PR-program is represented by a flow-chart we mean that a tree is given, where the vertices represent the various steps and carry the additional information which explains the nature of that step in coded form given by corresponding integers (referring to other vertices, e.g.). A PR-program is called *explicit* if m, n, q, c and the combinatorial structure in form of a flow-chart with the information described are given explicitly. A PR-program is called *certainly computable* (cc) if an effective procedure is given which works out the explicit form of the PR-program.

PROPOSITION 3. *Every (cc) PR-program calculates a (cc) PR-function and, conversely, every (cc) PR-function can be calculated by a suitable (cc) PR-program.*

Proof. In the first part of Proposition 3 we only have to justify the cc-version in case of an explicit program. But it is clear in this situation that all computations lead to cc rational functions in $\mathbb{Z}(x)$ so that the P-sets and the result functions are all cc.

To prove the converse we consider a decomposition of $\mathbb{C}^n$ into finitely many, disjoint P-sets $P_1, \ldots, P_M$ ($M \in \mathbb{N}$) such that on each P_j all coordinates of our PR-function are either not defined at all or are given by rational formulae with denominators $\neq 0$ on all of P_j. The definition of each P_j involves certain polynomials in $\mathbb{Z}[x]$, and we start the program by calculating all of these polynomials (for $1 \leq j \leq M - 1$). Next we introduce tests involving these polynomials in a fixed order described as follows: First test the conditions for P_1, but the first time one of these conditions is violated move over to the conditions for P_2 and so on until P_{M-1}. If a test for P_{M-1} is also violated we make no further tests. This explains completely the branching of the program, and we observe that the $x \in P_j$ ($j = 1, \ldots, M - 1$) will occur exactly in the branches which finish with the tests corresponding to P_j while the $x \in P_M$ are those which lead for each P_j ($1 \leq j \leq M - 1$) to a violation of at least one corresponding test. Since each branch represents a P-set by itself it follows that the corresponding P-sets form a refinement of the original P-partition, and we know exactly whether the branch should lead to the result "empty" or to certain rational functions which are well defined on this P-set. In the latter case it is easy to calculate these functions by a completion of our program. In case that the given PR-function is cc we have an effective procedure for obtaining the polynomials defining the P-sets P_j and for the corresponding rational formulae. This can be translated into effective procedures for making the computations and tests in the PR-program explicit. Finally the parameter vector c consists of the integral coefficients of all occurring polynomials. Q.E.D.

Looking back at the discussion above we like to emphasize the difference between the algebraic and the analytic point of view. Algebraically, a PR-function is described by certain P-sets (given by polynomial equations and inequalities) and certain rational expressions on these P-sets; in this context x could be taken as a vector of indeterminants. Analytically, we seek to evaluate the PR-function for each x of its domain by actual arithmetic calculations. The PR-programs explain exactly how this evaluation can be brought about (by an effective procedure in the cc situation).

10. Applications to Complexity Theory

Two situations in which cc PA-functions and cc PA-sets arise naturally are the determination of the rank of a tensor and the evaluation of polynomials. This will be explained by the results in this section.

DEFINITION. An *$(m_1, \dots, m_n)$-tensor* is a function T: $\{1, \dots, m_1\} \times \{1, \dots, m_2\} \times \cdots \times \{1, \dots, m_n\} \to \mathbb{C}$, where n and $m_1, \dots, m_n$ are fixed, natural numbers. We denote its values by $t_{j_1 \dots j_n}$. Its *rank* is the least number $R \in \mathbb{N}_0$ for which we can find a representation $t_{j_1 \dots j_n} = \sum_{r=1}^{R} a_{j_1}^{(1,r)} \dots a_{j_n}^{(n,r)}$ $(\forall j_1, \dots, j_n)$ with complex numbers $a_{j_k}^{(k,r)}$.

If we arrange the values $t_{j_1 \dots j_n}$ according to a prescribed linear ordering of the indices we can identify each $(m_1, \dots, m_n)$-tensor with a vector in $\mathbb{C}^m$, where $m = m_1 \cdot \dots \cdot m_n$. Then the rank can be interpreted as an integer-valued function "rank": $\mathbb{C}^m \to \mathbb{C}$, which is defined everywhere.

THEOREM 5. *The rank of an $(m_1, \dots, m_n)$-tensor is a cc piecewise constant function* ($\mathbb{C}^m - \to \mathbb{C}$).

Proof. For fixed $R \in \mathbb{N}_0$ the set given by all equations $t_{j_1 \dots j_n} = \sum_{r=1}^{R} a_{j_1}^{(1,r)} \dots a_{j_n}^{(n,r)}$ is a cc P-set in the variables $(t_{j_1 \dots j_n}, a_{k_1}^{(1,r)}, \dots, a_{k_n}^{(n,r)})$ of $\mathbb{C}^{m+mR}$. In case $R = 0$ it consists exactly of the zero-tensor while for $R \geq 1$ its projection onto the t-coordinates yields the set of tensors of rank at most R. Due to Lemma 4 this is a cc PA-set in $\mathbb{C}^m$. Eliminating the tensors of rank at most $R - 1$ leaves the set of tensors of rank R as the difference of two cc PA-sets; hence it is a cc PA-set by Remark 1. The observation that the rank cannot exceed m completes the proof. Q.E.D.

Theorem 5 applies in particular to the usual rank of a matrix ($n = 2$). Here, of course, the matrices of fixed rank are characterized by explicit equations and inequalities using subdeterminants.

An interesting consequence of Theorem 5 is that it yields an effective procedure to determine the nonscalar complexity of bilinear programs, e.g., the multiplication of rectangular matrices with fixed dimensions [2, pp. 41–43].

For the problem of polynomial evaluation we introduce the notion of a *polynomial program* which is closely related to the model of computation in [2, pp. 5 ff]. It is defined as a P-program without tests and divisions whose result is never "empty"; therefore it runs uniformly for all $x \in \mathbb{C}^n$. We denote by $N(\cdot)$ the number of multiplications and $N(\pm)$ the number of additions/subtractions; they are independent of x and yield $N_0 = N(\cdot) + N(\pm)$ for the computational complexity. We say that two programs have the *same structure* if they differ only by the value of the parameter vector c; this structure is called cc if the whole P-program is cc with the exception of the parameter vector c. Such a polynomial program computes the values of finitely many polynomials $p_j(x) \in \mathbb{C}[x]$ $(1 \le j \le m,\ m \in \mathbb{N})$. We can write $p_j = \sum_k a_{jk} x^k$ with $x^k = x_1^{k_1} \dots x_n^{k_n}$, where k runs through all vectors $(k_1, \dots, k_n) \in \mathbb{N}_0^n$ with $0 \le k_l \le d$ for some fixed $d \in \mathbb{N}$. By putting the indices (j, k) into some prescribed linear ordering we can identify the system $(p_1, \dots, p_m)$ with a coefficient vector $a \in \mathbb{C}^M$ with $M = m(d+1)^n$.

Now we are interested in finding for each choice of the coefficient vector a shortest polynomial program, i.e., with minimal computational complexity, for the computation of the corresponding polynomial system. We emphasize that the values of these polynomials are to be computed for all x.

THEOREM 6. *For given $m, n, d \in \mathbb{N}$ the space $\mathbb{C}^M$ divides into finitely many non-empty disjoint cc PA-sets A_k $(1 \le k \le K)$ such that all polynomial systems whose coefficient vectors belong to the same A_k can be computed by polynomial programs of the same cc structure and none of these polynomial systems can be computed by a shorter polynomial program.*

As a consequence, the computational complexity of our polynomial system is a cc piecewise constant function of a.

Proof. The iterated application of Horner's rule leads to programs of the same cc structure where the parameter vector c is the coefficient vector a and where the number of computations is $N_0 = 2(M - m)$. Hence we need only consider polynomial programs with $N_0 \le 2(M - m)$. Each computation involves two known quantities and one of the three operations $(+, -, \cdot)$. Thus the parameter vector c can always be taken from $\mathbb{C}^{4(M-m)}$ and then we deal only with finitely many different structures of polynomial programs which can be listed effectively. We will now show that all polynomial programs of such a fixed structure compute polynomial systems whose coefficients form a cc PA-set. Let us therefore consider all polynomial programs of a fixed structure. In these programs only c varies and if we successively insert the expressions for the known quantities occurring in the computations we see that these programs compute exactly a certain system of cc polynomials from $\mathbb{Z}[c, x]$. Thus we can compare coefficients and learn that the a_{jk} must equal certain cc polynomials from $\mathbb{Z}[c]$. In this characteristic way a cc PA-relation in (a, c) is associated with the given structure. Then we can determine the polynomial systems that are computed by all polynomial programs of this structure by simply projecting this PA-relation into the space $\mathbb{C}^M$ of coefficient vectors which leads to a cc PA-set $A \subseteq \mathbb{C}^M$. Now we start by successively determining these cc PA-sets for our programs with $N_0 = 0$, then for programs with $N_0 = 1$ and so on; but for each new PA-set we delete all the coefficient vectors belonging to PA-sets that were already constructed. This can be done following the procedure in Remark 1 and yields finitely many, disjoint, cc PA-sets with the desired properties. Q.E.D.

The proof shows that for every polynomial system we can find a shortest polynomial program whose structure can be effectively determined and whose parameter vector is obtained by solving cc algebraic equations over $\mathbb{Z}[a]$. This last step is usually called algebraic preconditioning and may be computationally involved in a concrete case, although it may be worth the effort since the program is to be applied for all x. However, it might be natural to require that the parameter vector must be rational in the coefficients, i.e., $c \in \mathbb{Z}(a)$. This is usually called rational preconditioning, and Theorem 6 holds for programs of this type as well if we restrict the permitted rational functions $(a - \to c)$ to a fixed finite set of rational formulae which must include the case $c = a$ (Horner's rule).

Remark 7. Theorem 6 remains true if we allow the polynomials p_j to have different degrees for each variable which may also depend on j. For that purpose we choose d large enough and require some coefficients to be zero.

It is also possible to incorporate division as a possible operation. But in this case we have the situation that we deal with quotients of polynomials from $\mathbb{Z}[c, x]$ and therefore must demand that the denominators do not vanish. Hence the program may not work for certain values of the variables.

It is also possible to prove analogous results for $N(\pm)$ (resp. $N(\cdot)$) instead of N_0 in the same way by using the canonical forms of the programs given in [2, pp. 60 ff] when we assume $N(\cdot) \le h$ (resp. $N(\pm) \le h$) for an arbitrary integer $h \ge M - m$.

An interesting consequence of Theorem 6 concerns the dimensions of the involved sets. Since the space $\mathbb{C}^M$ has dimension M, exactly one of the PA-sets A_k must have dimension M, too. By Remark 6 this cc set A_k therefore contains a free cc P-set A because any decomposition of A_k into P-sets contains a free P-set. Furthermore, the complement of A has dimension at most $M - 1$ by Remark 6. This shows that *there is a fixed structure such that all of the polynomial systems under consideration can be computed by polynomial programs of this structure with a minimal number of computations unless their coefficients satisfy a certain non-trivial polynomial equation*. Furthermore, we have given an effective procedure to determine one possible equation and a corresponding structure. For one polynomial in one variable they were determined in [10].

When we restrict our attention to additions and subtractions then we can completely determine the connection between the dimension and the complexity $N(\pm)$. For that purpose we again consider a fixed number m ($\in \mathbb{N}$) of polynomials in n ($\in \mathbb{N}$) variables. Furthermore, two integers $N \geq 0$ and $h \geq N$ should be given, and we assume that the degree d is such that $M \geq N + m$.

THEOREM 7. *For given m, n, d and N, h the coefficients of those polynomial systems that can be computed by polynomial programs with $N(\pm) = N$, $N(\cdot) \leq h$ form a cc PA-set of dimension $N + m$ in $\mathbb{C}^M$.*

Proof. From the proof of Theorem 6 we know that we encounter only finitely many structures of polynomial programs. Furthermore, the coefficients of the polynomial systems that are computed by the programs of a fixed structure form a cc PA-set. Hence we need only determine the dimension of this set. For that purpose we fix a structure with $N(\pm) = N$ and the parameter vector c.

First we transform our program into an additive canonical form according to [2, p. 61] so that in each additive step only one constant occurs (which can be determined from (c)) while the other possible constant can be introduced later as a factor. In this way we find $N + m$ new constants which make up a parameter vector $\tilde{c}$ of length $N + m$ and a corresponding polynomial program which calculates the same polynomial system. Hence the coefficient vector a is represented by polynomials in $\mathbb{Z}[\tilde{c}]$ and has transcendency degree at most $N + m$ over $\mathbb{Q}$. It follows that the dimension of our PA-set is at most $N + m$.

On the other hand, we order the indices j, k ($1 \leq j \leq m, 0 \leq k_l \leq d$ for $l = 1, \ldots, n$) lexicographically, namely first with respect to k_n, then with respect to k_{n-1} and so on until, finally, with respect to j. Then we consider a polynomial system whose smallest (in this ordering) $N + m$ ($\leq M$) coefficients a_{jk} are algebraically independent over $\mathbb{Q}$ whereas all the others are zero. This is possible due to our assumption about the degree d, and this polynomial system can be computed by iterated application of Horner's rule with $N(\cdot) = N(\pm) = N \leq h$. Hence it belongs to the examined PA-set whose dimension therefore is at least $N + m$. Q.E.D.

Remark 8. Suppose that $h \geq M - m$ and consider for given $N \in \mathbb{N}_0$ the coefficients of those polynomial systems that can be computed by polynomial programs with $N(\pm) = N$ and $N(\cdot) \leq h$, but not by any such program with $N(\pm) < N$ and $N(\cdot) \leq h$. These are the polynomial systems with additive complexity N under the restriction $N(\cdot) \leq h$. From Horner's rule we know that the possible values of N range over $0 \leq N \leq M - m$, and from Theorem 7 we infer that *the coefficient vectors a yielding the same complexity N form a cc PA-set of dimension* $N + m$. In particular, the highest complexity $N = M - m$ occurs on a cc PA-set of dimension M. It follows that the restricted additive complexity equals $M - m$ unless the coefficients a satisfy a non-trivial polynomial equation which can be found by an effective procedure. If the coefficients are algebraically independent they do not satisfy any non-trivial polynomial equation. Hence also the unrestricted additive complexity equals $M - m$ (arbitrary $N(\cdot)$). For $m = 1$ this is Belaga's theorem [2, p. 61].

11. Applications to Linear Algebra

In this section we will show that the solutions of three important problems in linear algebra can be given in terms of cc PA-functions (resp. cc PR-functions). For a complex (n, n)-matrix A and a vector $b \in \mathbb{C}^n$ we want to find

—the solutions x of $Ax = b$,

—a Jordan canonical form J of A and a corresponding similarity transformation T,

—a PLU-decomposition of A.

Throughout this section $n \in \mathbb{N}$ is fixed and (n, n)-matrices are identified with vectors in $\mathbb{C}^{n^2}$.

Let us first discuss the solution of a system of linear equations. Obviously, the equation $Ax - b = 0$ defines a cc PA-relation in (A, b) and x. But we can also describe all solutions x—if they exist—in a parametrized form, i.e., as the totality of vectors $x_0 + B\lambda$ ($\lambda \in \mathbb{C}^n$) with a certain (n, n)-matrix B and a vector $x_0 \in \mathbb{C}^n$. We call (B, x_0) a *parametrization* for (A, b).

Proposition 4 (Gauss' elimination). *There exists a cc PR-function* ($\mathbb{C}^{(n+1)n} - \to \mathbb{C}^{(n+1)n}$) *which associates a parametrization* (B, x_0) *with every pair* (A, b) *in the case that* $Ax = b$ *is solvable.*

Proof. According to Proposition 3 it suffices to observe that the Gauss' elimination is a cc PR-program with parameter vector $(0, 1)$ for the computation of x_0 and B (which can be brought into square form by appending zero-columns). For that purpose we use the algorithm in [5, pp. 202 ff]. Besides arithmetic operations it involves testing in a definite order whether certain matrix elements are zero or not and permutations which can be expressed via multiplications by 1. The cases without solution (result "empty") are clear on the basis of such tests, and in all other cases the vector x_0 and the matrix B can be read off directly from the computed quantities apart from additional zeros. Q.E.D.

Notice that the present information in each step of the program tells us exactly where we generated ones and zeros and therefore the permutations could be omitted. At the result step the present information describes a P-set in (A, b) and, in case that the result is not "empty," the relation between (A, b) and x is normal. Thus *the algorithm generates a normal decomposition of our relation*.

Proposition 4 can also be applied to rectangular matrices by interpreting them as square matrices where certain elements are zero. In this way we obtain another procedure to determine the rank of a matrix (cf. Theorem 5).

Let us now turn to the computation of the Jordan canonical form. A matrix J is said to be a *Jordan matrix* if $J = \operatorname{diag}(J_1, \dots, J_r)$ with (Jordan) blocks J_k of the form

$$\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \qquad (\lambda \in \mathbb{C}).$$

The dimensions n_k of the blocks J_k determine the *block structure* $(n_1, \dots, n_r)$; and conversely each such r-tupel (for any $r \in \mathbb{N}$) occurs as a block structure provided that $n_k \geq 1$ $(\forall k)$ and $n_1 + \cdots + n_r = n$.

For a given matrix A we want to find such a Jordan matrix J and an invertible T satisfying $T^{-1}AT = J$; in this case we call J a Jordan canonical form of A.

THEOREM 8 (Jordan canonical form). *The conditions $T^{-1}AT = J$, J Jordan matrix, T invertible define a cc PA-relation in A and (J, T). If we associate with every matrix A all complete systems $\lambda \in \mathbb{C}^n$ of eigenvalues (with algebraic multiplicity) we obtain a cc PA-function $(\mathbb{C}^{n^2} - \to \mathbb{C}^n)$. Furthermore there exists a cc piecewise linear function $(\mathbb{C}^{n^2+n} - \to \mathbb{C}^{n^2})$ which computes for every pair (A, λ), where λ is a complete system of eigenvalues of A, a corresponding Jordan matrix J. Finally there is a cc PR-function $(\mathbb{C}^{2n^2} - \to \mathbb{C}^{n^2})$ which associates with each pair (A, J), where J is possible Jordan canonical form of A, an invertible T satisfying $T^{-1}AT = J$.*

Proof. Let us first consider Jordan matrices of a fixed block structure $(n_1, \dots, n_r)$. They can be described among all matrices by the requirements $\lambda_1 = \cdots = \lambda_{n_1}$, $\lambda_{n_1+1} = \cdots = \lambda_{n_1+n_2}$, etc. on the diagonal, together with the corresponding location of the ones and zeros off the diagonal. Combining this description with the equation $AT = TJ$ and the inequality $\det T \neq 0$ shows that the triplets (A, J, T), where J has fixed block structure, form a cc PA-relation. Since there are only finitely many possible block structures, this proves the first claim.

The complete systems of eigenvalues of A are exactly the complete systems of solutions of the characteristic polynomial of A, whose coefficients are polynomials in the elements of A that can easily be computed. This situation gives rise to a cc PA-function ($\mathbb{C}^{n^2} - \rightarrow \mathbb{C}^n$) according to Lemma 3. Therefore the pairs (A, λ), where λ runs through all the complete systems of eigenvalues of A, form a cc PA-set in $\mathbb{C}^{n^2+n}$. Using finitely many tests we can determine which of the eigenvalues in λ are equal and which are not. Each eigenvalue λ_j $(1 \leq j \leq n)$ occurs in certain Jordan blocks whose dimensions can be found according to the fact that $\operatorname{rank}(A - \lambda_j I)^{k-1} - \operatorname{rank}(A - \lambda_j I)^k$, for $k \in \mathbb{N}$, is the number of Jordan blocks of dimension at least k corresponding to the eigenvalue λ_j (notice that $k \leq n$ suffices). In this way we determine for each index j $(1 \leq j \leq n)$ all the indices k with $\lambda_j = \lambda_k$ and the dimensions of the Jordan blocks (including their number) corresponding to λ_j. Pairs (A, λ) for which these informations are the same are collected in one and the same set; these sets are then cc PA-sets due to Theorem 5. For all pairs belonging to the same set we can now easily construct a corresponding Jordan matrix J by prescribing the order of the indices according to which the eigenvalues λ_j occur on the diagonal of J and the corresponding block structure. Thus we have an explicit procedure to determine all the elements of J; since they are 0, 1, or a λ_j, this yields a cc piecewise linear function.

Finally, we consider all the pairs (A, J), where J is a Jordan matrix corresponding to A. They are obtained as projections from the triplets (A, J, T) and hence form a cc PA-set in $\mathbb{C}^{2n^2}$. The possible transformations T satisfy $AT = TJ$ which is a system of homogeneous linear equations for the elements of T. Therefore, Proposition 4 applies and yields $T = \sum_{k=1}^{n^2} t_k T_k$, where the t_k are arbitrary complex numbers and T_k $(1 \leq k \leq n^2)$ are cc PR-functions in (A, J). Thus we are done when we can determine, e.g., integer values $t_1, \ldots, t_{n^2}$ such that $\det T \neq 0$. Now $\det T$ is a polynomial in $t_1, \ldots, t_{n^2}$ whose coefficients are cc PR-functions in (A, J) and which has degree at most n in each t_k. We allow each t_k to assume any of the $n + 1$ values $0, 1, \ldots, n$ and check in each of these $(n + 1)^{n^2}$ cases the polynomial condition $\det T \neq 0$. This defines a cc PA-subset in (A, J), and if this set is non-empty we also have an admissible transformation T. This effective procedure leads after separation to the claimed PR-function provided that we can show that $\det T \neq 0$ happens at least for one of our choices. Assume that for a certain pair (A, J) we always find $\det T = 0$. Since we inserted for each t_k more values than its degree, it is clear that $\det T$ vanishes identically in the variables $t_1, \ldots, t_{n^2}$. But this is a contradiction to the fact that for each pair (A, J) at least one admissible transformation exists. Q.E.D.

Notice that Theorem 8 yields a cc PA-function ($\mathbb{C}^{n^2} - \to \mathbb{C}^{2n^2}$) which computes for every A some pairs (J, T) with $T^{-1}AT = J$; this can be seen by combining the individual functions described in the theorem.

The third and last result in this section deals with triangular decompositions of matrices.

PROPOSITION 5 (PLU decomposition). *There exists a cc PR-function* ($\mathbb{C}^{n^2} - \to \mathbb{C}^{3n^2}$) *which computes for every (n, n)-matrix A three matrices P, L, U satisfying $A = PLU$, where P is a permutation matrix, L is a lower triangular matrix with* diag $L = I$, *and U is an upper triangular matrix.*

Proof. The matrices P, L, U can be determined by a Gauss' elimination process described in [12, pp. 134 ff] which is very similar to the proof of Proposition 4, except that we also keep track of the permutations and elementary row operations. It is clear that this can be phrased as a cc PR-program with parameter vector $(0, 1)$ and thus yields the result. Q.E.D.

Remark 9. The same result is true for PUL decompositions where $\operatorname{diag} U = I$ should hold; for that purpose we start by working on the last column upwards from the bottom. By transposition LUP or ULP decompositions can be determined by cc PR-programs, where each time the triangular matrix occurring in the middle should have all diagonal elements equal to one.

12. APPLICATIONS TO MEROMORPHIC DIFFERENTIAL EQUATIONS

This final section is used to explain that PA-functions arise in connection with the differential equation $X'(z) = A(z)X(z)$, where $zA(z)$ is a polynomial (n, n)-matrix in z of degree r ($n \geq 2$, $r \geq 1$ are fixed throughout this section) and $X(z)$ is a fundamental solution matrix. It turns out that the essential parts of the formal solutions and certain algebraic invariants are PR-functions of the coefficients in $A(z)$ provided that $A(z)$ represents a rather general standard situation. Further and more essential applications will be given in [7]. For a detailed discussion of meromorphic differential equations the reader may consult [16].

The matrix $A(z)$ has $r + 1$ coefficient matrices whose elements are considered as the parameters of the equation. By arranging them in a prescribed way we associate with $A(z)$ a vector in $\mathbb{C}^{(r+1)n^2}$ which is restricted by the requirement that the leading coefficient matrix is not the zero matrix.

Now let us first investigate the formal solutions at ∞. We require that one of them should have the form $F(z)z^{\Lambda}e^{Q(z)}$, where $Q(z) = \operatorname{diag}[q_j(z)]$ is a polynomial matrix of degree r with $Q(0) = O$, $\Lambda = \operatorname{diag}[\lambda_j]$ is a constant matrix satisfying

$$\lambda_j \not\equiv \lambda_k \pmod 1 \qquad \text{if } q_j = q_k \text{ holds } (j \neq k), \tag{$*$}$$

and $F(z) = I + \sum_{k=1}^{\infty} F_k z^{-k}$ is a formal series (compare to [16, p. 111]). If $X' = AX$ has such a formal fundamental solution we call $A(z)$ *admissible*. The parameter vectors of the admissible matrices form a subset $\mathscr{A} = \mathscr{A}(n, r)$ of $\mathbb{C}^{(r+1)n^2}$. In general, formal fundamental solutions can be more complicated, and they are only determined up to a constant invertible matrix factor on the right. But if there exists such a simple formal solution as described above this formal solution is uniquely determined [7]. Thus the admissible $A(z)$ form an especially interesting class, and it is an important problem to characterize this class and to find formulae for the unique matrices Q, Λ, F. It is interesting to note that $\mathscr{A}$ is no PA-set, since otherwise the part of $\mathscr{A}$, where A is diagonal and Q is scalar, would be a PA-set as well, but here the incongruence condition $(*)$ implies the opposite. Hence we must generalize our concepts.

DEFINITION. A set $M \subseteq \mathbb{C}^s$, $s \in \mathbb{N}$, is called a (cc) PA-set *relative to* $\mathscr{A} (\subseteq \mathbb{C}^s)$ if it can be written as $M = M' \cap \mathscr{A}$ with a (cc) PA-set M' (and an effective procedure for finding M'). Accordingly, we talk about (cc) PA-relations (resp. PA-functions, resp. PR-functions) *relative to* $\mathscr{A}$ if they are obtained from (cc) PA-relations (resp. PA-functions, resp. PR-functions) by restricting the preimage variable to $\mathscr{A}$.

The following result will be very useful.

PROPOSITION 6. *Suppose that we have a (cc) PA-relation which can be viewed as a single-valued function if we restrict the preimage variable to the set $\mathscr{A}$. Then this restriction can be represented by a (cc) PR-function relative to $\mathscr{A}$.*

This follows from the Cardinality Theorem if we consider the total part of the relation where the cardinality function is zero or one. By Remark 5 this part of the relation is the graph of a (cc) PR-function, and we obtain the result by restricting the preimage variable to $\mathscr{A}$.

In order to formulate the solution of our problem we represent $Q(z)$ by its coefficients so that (Q, Λ) is described by a vector in $\mathbb{C}^{(r+1)n}$ and each F_k is described by a vector in $\mathbb{C}^{n^2}$. For the moment $Q(z)$ is any diagonal polynomial matrix of degree r without constant term and Λ is any diagonal constant matrix; furthermore, $zA(z)$ is any polynomial matrix of degree r. Using these notations we have

THEOREM 9. *There exists a cc PR-function* $(A - \to (Q, \Lambda))$ *with the following properties*:

(i) *on its domain* $zA(z)$ *and* $Q(z)$ *have degree* r;

(ii) *if we restrict the domain by the incongruence condition* $(*)$ *we obtain exactly our set* $\mathscr{A}$;

(iii) *on* $\mathscr{A}$ *the matrices* Q, Λ *are the unique matrices of our formal solution.*

Furthermore, for each $k \in \mathbb{N}$ *there is a cc PR-function* $(A - \to F_k)$ *which gives the unique matrix* F_k *of our formal solution for all* $A \in \mathscr{A}$; *in other words, our* F_k *are represented by cc PR-functions relative to* $\mathscr{A}$.

Proof. Insertion of the formal solution into the differential equation shows that it satisfies the condition $AF = F(Q' + \Lambda z^{-1}) + F'$. If we replace F by the truncated $\tilde{F} = I + \sum_{k=1}^{m+r} F_k z^{-k}$, $m \in \mathbb{N}_0$, we see that $A\tilde{F}$ and $\tilde{F}(Q' + \Lambda z^{-1}) + \tilde{F}'$ agree down to terms of order z^{-m-1}.

Now we consider for fixed $m \in \mathbb{N}_0$ all A, Q, Λ (as specified before Theorem 9) and all $\tilde{F} = I + \sum_{k=1}^{m+r} \tilde{F}_k z^{-k}$ (represented by vectors in $\mathbb{C}^{(m+r)n^2}$) with the property that $A\tilde{F}$ and $\tilde{F}(Q' + \Lambda z^{-1}) + \tilde{F}'$ agree down to terms of order z^{-m-1}. This clearly defines a cc PA-relation in A and $(Q, \Lambda, \tilde{F})$ by equating certain coefficients. Since truncation of $\tilde{F}$ leads to an element of a relation with smaller m, we observe that an increase of m results in further restrictions for A, Q, Λ.

First we discuss the case $m = 0$, where the requirement is that A agrees with $\tilde{F}(Q' + \Lambda z^{-1})\tilde{F}^{-1}$ down to terms of order z^{-1}. If we denote $Q' + \Lambda/z$ temporarily by $D = \operatorname{diag}[d_j]$ we see that two possible choices D and $\hat{D}$ are related by the condition that $\hat{F}D$ and $\hat{D}\hat{F}$ agree down to terms of order z^{-1}, where $\hat{F}$ is a certain formal series of the form $I + \sum_{k=1}^{\infty} \hat{F}_k z^{-k}$. Concerning the diagonal terms $\hat{f}_{jj} d_j$ and $\hat{d}_j \hat{f}_{jj}$ we see that d_j and $\hat{d}_j$ must agree down to terms of order z^{-1} and, hence, are identical. It follows that in our relation between A and $(Q, \Lambda, \tilde{F})$ the matrices Q, Λ are uniquely determined by A. The relation in A and (Q, Λ) obtained by projection therefore defines a cc PR-function $(A - \to (Q, \Lambda))$ according to Remark 5, and this will be the cc PR-function mentioned in the first part of Theorem 9. It satisfies condition (i) by construction. We also observe that for $m > 0$ the corresponding relation in A and (Q, Λ) is only a restriction of the relation just obtained for $m = 0$.

We denote by $\mathscr{A}_m$, $m \in \mathbb{N}_0$, the set of all A which occur in the relation corresponding to m between A and $(Q, \Lambda, \tilde{F})$, if we restrict A additionally by the incongruence condition $(*)$, where we use the cc PR-function constructed before. Clearly the $\mathscr{A}_m$ form a decreasing sequence of sets, all of which contain our $\mathscr{A}$. If $A \in \mathscr{A}_m$ we form $B(z) = \tilde{F}^{-1}A\tilde{F} - \tilde{F}^{-1}\tilde{F}'$

$= Q' + \Lambda z^{-1} + \Sigma_{k=m+2}^{\infty} B_k z^{-k}$, and we know from the general theory [16, pp. 52–54, 100–101, 20–25] that the differential equation $Y' = BY$ can formally be solved by $Y = (I + \Sigma_{k=m+1}^{\infty} \hat{F}_k z^{-k}) z^{\Lambda} e^{Q(z)}$. Thereby we obtain a formal solution $X = \tilde{F}Y$ of the required form for the original differential equation $X' = AX$. In particular, we see that $A \in \mathscr{A}$; hence all $\mathscr{A}_m$ equal $\mathscr{A}$. The case $m = 0$ proves property (ii) of Theorem 9. Furthermore, Q and Λ are the unique matrices of our formal solution, as stated in property (iii). Finally, in our unique formal solution F is given by $\tilde{F}(I + \Sigma_{k=m+1}^{\infty} \hat{F}_k z^{-k})$, so that $\tilde{F}_k = F_k$ for $k = 1, \ldots, m$. Hence we see that these matrices $\tilde{F}_k$ which occur in the cc PA-relation corresponding to m are uniquely determined by A if we restrict A to $\mathscr{A}_m = \mathscr{A}$. By Proposition 6 it now follows that each F_k can be represented by a cc PR-function relative to $\mathscr{A}$. Q.E.D.

Remark 10. When we are interested in the computation of an actual solution of $X' = AX$ at 0 then we replace the requirement that A should be admissible by the condition that no two eigenvalues of the coefficient of z^{-1} in $A(z)$—called A_0—differ by a positive integer. Then the discussion in [16, pp. 18–20] shows that the differential equation has a unique solution of the form $(I + \Sigma_{k=1}^{\infty} P_k z^k) z^{A_0}$ and that the n^2 elements of each P_k are obtained by solving n^2 linear equations with unique solution and hence are cc rational functions (with integer coefficients) in the parameters of A. If in addition a complete system $\lambda \in \mathbb{C}^n$ of eigenvalues of A_0 is known then we can apply Theorem 8 to find an certain Jordan matrix M and a certain invertible matrix T with $A_0 = TMT^{-1}$. Thus we find a certain solution $(\Sigma_{k=0}^{\infty} E_k z^k) z^M$, where the elements of each $E_k = P_k T$ are cc PR-functions in (A, λ) relative to the set of those (A, λ) that satisfy the condition imposed on A_0. In this case the solution is not unique per se but depends on the choice made in the computation of T. All the other solutions of this kind with the same M are obtained as $(\Sigma_{k=0}^{\infty} E_k C z^k) z^M$, where C can be any invertible matrix that commutes with M. Since other Jordan canonical forms of A_0 are given as block permutations of M, i.e., $P^{-1}MP = \tilde{M}$, a corresponding solution is given by $(\Sigma_{k=0}^{\infty} E_k P z^k) z^{\tilde{M}}$.

Our final example is the normalization of our $A(z)$ by means of constant (in z), invertible, diagonal matrices D. If for $A(z), B(z)$ such a D exists satisfying $B = D^{-1}AD$ we call A and B *equivalent* ($A \sim B$). This defines an equivalence relation on the set of matrices $A(z)$ which have the property that $zA(z)$ is a polynomial matrix of degree r. When we choose a system of representatives under this equivalence we call the representatives *normalized* matrices.

PROPOSITION 7. *A system of representatives can be chosen in such a way, that the function which maps every matrix $A(z)$ onto its representative is a cc PR − function* $(\mathbb{C}^{(r+1)n^2} - \to \mathbb{C}^{(r+1)n^2})$.

Proof. We describe an effective procedure which yields a PR-program for the computation of a representative. For that purpose, we associate with $A(z)$ a graph whose vertices are the numbers $1, \dots, n$ and where $j \neq k$ are joined by an edge if and only if $a_{jk}(z) \not\equiv 0$ or $a_{kj}(z) \not\equiv 0$. Notice that such a condition could be checked by testing whether the coefficients of the function vanish starting with the coefficient of z^{r-1}. There are only finitely many graphs and for each graph on n vertices we describe how to compute the matrix D which transforms $A(z)$ into its representative $D^{-1}AD = [a_{jk}d_k/d_j]$.

Each graph decomposes into its connected components (which correspond to a diagonal blocking of A), and for each component we think of having a priori selected a spanning tree [9, p. 40]. If the edge between j and k ($j < k$) belongs to a spanning tree then we require that $a_{jk}d_k/d_j$ is monic if $a_{jk} \not\equiv 0$ or otherwise that $a_{kj}d_j/d_k$ is monic (leading coefficient one). This can be done effectively, since a tree has no cycles and determines the d_k (for all k in one component) uniquely up to an arbitrary multiplicative factor ($\neq 0$) because the spanning tree is connected and contains all vertices of the component. The choice of this factor does not influence the resulting matrix $D^{-1}AD$ and we may therefore choose $d_j = 1$ for the least index j in the component.

All the combinatorial choices can be made effectively in a definite way and result in a cc PR-program with parameter vector $(0, 1)$ for the computation of D and $\tilde{A} = D^{-1}AD$.

We observe that the constructed matrix $\tilde{A}$ has the property that for each edge between j and k ($j < k$) of our spanning trees $\tilde{a}_{jk}$ is monic if $\tilde{a}_{jk} \not\equiv 0$ or otherwise $\tilde{a}_{kj}$ is monic. Now suppose that A and B are equivalent. It follows that $\tilde{A}$ and $\tilde{B}$ are equivalent also, i.e., $\tilde{B} = D^{-1}\tilde{A}D$; moreover, the respective graphs and selected trees coincide. Using our previous arguments we see that d_j must be the same along a component. In view of the block structure of $\tilde{A}$ the matrix D commutes with $\tilde{A}$ so that $\tilde{B} = \tilde{A}$. So $\tilde{A}$ is a unique representative of the equivalence class of A. Q.E.D.

We see that the pairs (A, B) with $A \sim B$ form a cc PA-relation in $\mathbb{C}^{(r+1)n^2} \times \mathbb{C}^{(r+1)n^2}$ because this set is the projection of the cc PA-set $\{(A, B, D) : DB = AD$, where A, B, D are of the permitted types$\}$.

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